MATH 579 Exam 6 Solutions

Part I: Let p, q be n-permutations of the same type (i.e. the same number of cycles of each length). Prove that there is some n-permutation r such that $p = rqr^{-1}$. For example, with $p = (1532)(46), q = (1234)(56)$, we can take $r = (254)$, and $(1532)(46) = (254)(1234)(56)(452)$.

Put p, q into partition form and read off the numbers in order; p is $p_1, p_2, \ldots, p_n$ with parentheses inserted in various places; similarly q is $q_1, q_2, \ldots q_n$. [In the example $p_1 = 1, p_2 = 5, p_3 = 3, p_4 = 2, p_5 = 4, p_6 = 6$]. Define r via $r(q_i) = p_i$; hence $r^{-1}(p_i) = q_i$. Let $p_i \in [n]$. The proof splits into two cases; if p_i is not at the end of a cycle in p, then $p(p_i) = p_{i+1}$. Because p, q have the same type $q(q_i) = q_{i+1}$. Now $rqr^{-1}(p_i) = rq(q_i) = r(q_{i+1}) = p_{i+1} = p(p_i)$. Alternatively, if p_i is at the end of a cycle in p, then $p(p_i) = p_j$, for some $j < i$, but also $q(q_i) = q_j$. So, $rqr^{-1}(p_i) = rq(q_i) = r(q_j) = p_j = p(p_i)$. Hence, p and rqr^{-1} agree on each element of $[n]$ and are equal.

Part II:

1. How many permutations $p \in S_4$ satisfy $p^2 = 1$?

   Such a permutation must have all cycles of length either 2 or 1 (length must divide 2). Hence, either two two-cycles, one two-cycle and two fixed points, or four fixed points. There are $\frac{4!}{2^2 2!} = 3$, $\frac{4!}{1^2 2^1 2!1!} = 6$, $\frac{4!}{1^4 4!} = 1$ types, respectively. Hence, altogether there are 10. Alternatively, one may list them: $(12)(34), (13)(24), (14)(23), (12), (13), (14), (23), (24), (34), 1$.

2. A permutation p is called an involution if $p^2 = 1$. Prove that for $n > 1$, the number of involutions in S_n is even.

   There are $n!$ permutations altogether, which is even for $n > 1$. We take away the non-involutions and leave only involutions. We now prove that there are an even number of non-involutions, which solves the problem because the difference between two even numbers is even. Every non-involution p may be paired off with its inverse p^{-1}. If they were the same, then $1 = pp^{-1} = p^2$, so p would be an involution, but p was a non-involution. Hence each pair contains two distinct permutations.

3. Let n be even. Prove that $c(n, n/2) \geq \frac{n!}{2^{n/2}(n/2)!}$.

   $c(n, n/2)$ counts the number of permutations of $[n]$ with exactly $n/2$ cycles. By Thm. 6.9, $\frac{n!}{2^{n/2}(n/2)!}$ counts the number of permutations of $[n]$ with exactly $n/2$ cycles, each of length 2; this is a subset of what is being counted by $c(n, n/2)$; in fact for $n > 2$ it is a proper subset.

4. Let $n \geq 3$. How many n-permutations have $1, 2, 3$ in the same cycle?

   The question does not depend on the specific identities of $1, 2, 3$; so without loss we consider $n, n-1, n-2$ instead. Consider the Bona form of an n-permutation. $n, n-1, n-2$ are in the same cycle precisely when n comes before both $n-1, n-2$. Just one-third of all $n!$ permutations have this property.

5. Let $n \geq 3$. How many n-permutations have 1 in the same cycle with either 2 or 3, but not both?

   As in the previous problem, we consider instead $n, n-1, n-2$ and use Bona form. n is in the same cycle with either $n-1$ or $n-2$ precisely when n comes between them, i.e. $\ldots, n-1, \ldots, n, \ldots, n-2, \ldots$ or $\ldots, n-2, \ldots, n, \ldots, n-1, \ldots$. Hence again one-third of all $n!$ permutations have this property.

Exam grades: High score=104, Median score=76, Low score=54